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# A Formula for the Characteristic Polynomial of an Arrangement

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## 1. INTRODUCTION

Let $\mathbf{K}$ be a field and let $V$ be a vector space of dimension $l$ over $\mathbf{K}$. A hyperplane in $V$ is a vector subspace of codimension 1. An arrangement $\mathscr{A}$ in $V$ is a finite set of hyperplanes. Let $L(\mathscr{A})$ be the collection of all intersections of elements of $\mathscr{A}$. We partially order $L(\mathscr{A})$ by the reverse of inclusion, so that $X \leqslant Y$ means $X \supseteq Y$. Then $L(\mathscr{A})$ is a geometric lattice, called the intersection lattice of $\mathscr{A}$, which has $V$ as its minimal element. Let $\mu$ be the Möbius function [14, p. 342] of $L(\mathscr{A})$. The *characteristic polynomial* $\chi(\mathscr{A}; t)$ of $\mathscr{A}$ is defined by

$$\chi(\mathscr{A}; t) = \sum_{X \in L(\mathscr{A})} \mu(V, X)\, t^{\dim X}. \tag{1.1}$$

If $\bigcap_{H \in \mathscr{A}} H = \{0\}$, then $\chi(\mathscr{A}; t)$ is the characteristic polynomial [9, p. 128] of $L(\mathscr{A})$. In [11] Terao showed that if $\mathscr{A}$ is a free arrangement then the characteristic polynomial factors in $\mathbf{Z}[t]$ as a product of linear factors. Its non-negative integer roots are the degrees of certain derivations of $\mathbf{K}[x_1, \ldots, x_l]$. In this paper we introduce a family of graded modules $D^p(\mathscr{A})$, $p \geqslant 0$, and show that the characteristic polynomial of *any* arrangement may be computed in terms of the Poincaré series of the modules $D^p(\mathscr{A})$. Our main result is Theorem (1.2). In case $\mathscr{A}$ is a free

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arrangement the factorization theorem is an easy consequence of Theorem (1.2).

We sketch the definition of the graded modules $D^p(\mathscr{A})$. Let $S = S(V^*)$ be the symmetric algebra of the dual space $V^*$ of $V$. Regard $S$ as a graded $\mathbf{K}$-algebra in which $\deg f = 1$ for all $f \in V^* \backslash \{0\}$. Let $p \geq 0$. Let $\mathrm{Der}^p(S)$ be the $S$-module of all alternating $p$-linear functions $\theta: S \times \dots \times S \to S$ which are $\mathbf{K}$-derivations in each variable. Then $\mathrm{Der}^p(S)$ is naturally a graded $S$-module. Note that $\mathrm{Der}^1(S) = \mathrm{Der}(S)$ is the module of $\mathbf{K}$-derivations of $S$. We agree that $\mathrm{Der}^0(S) = S$. Choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$ and let $Q = Q_{\mathscr{A}} = \prod_{H \in \mathscr{A}} \alpha_H \in S$. We call $Q$ a defining equation of $\mathscr{A}$. For $p \geq 1$, define

$$D^p(\mathscr{A}) = \{\theta \in \mathrm{Der}^p(S) \mid \theta(Q, f_2, \dots, f_p) \in QS \text{ for all } f_2, \dots, f_p \in S\}.$$

Let $D^0(\mathscr{A}) = S$. Then $D^p(\mathscr{A})$ is a graded $S$-submodule of $\mathrm{Der}^p(S)$. Let $\mathrm{Poin}(D^p(\mathscr{A}); x)$ be the Poincaré series of $D^p(\mathscr{A})$. Our main theorem is

(1.2) THEOREM. *For any arrangement $\mathscr{A}$ in $V$,*

$$\chi(\mathscr{A}; t) = (-1)^l \lim_{x \to 1} \sum_{p \geq 0} \mathrm{Poin}(D^p(\mathscr{A}); x)(t(x-1)-1)^p.$$

For simplicity we define

$$\Psi(\mathscr{A}; x, t) = \sum_{p \geq 0} \mathrm{Poin}(D^p(\mathscr{A}); x)(t(x-1)-1)^p.$$

Then we can rewrite the main Theorem (1.2) as

$$\chi(\mathscr{A}; t) = (-1)^l \lim_{x \to 1} \Psi(\mathscr{A}; x, t). \tag{1.3}$$

In Section 2, we prove general properties of $D^p(\mathscr{A})$. In Section 3, we consider the special case of a free arrangement. If $\mathscr{A}$ is free, then we show easily that

$$\Psi(\mathscr{A}; x, t) = \prod_{i=1}^{l} (1 + x + x^2 + \dots + x^{b_i - 1} - x^{b_i} t),$$

where $b_1, \dots, b_l$ are the exponents of $\mathscr{A}$. It follows from the main Theorem (1.3) that if $\mathscr{A}$ is free, then

$$\chi(\mathscr{A}; t) = \prod_{i=1}^{l} (t - b_i).$$

This factorization theorem was proved in [11] when $\mathbf{K} = \mathbf{C}$. Our proof of the main Theorem (1.3) begins in Section 4, where we introduce a family of complexes

$$D^{\cdot}(\mathscr{A})\colon 0 \to D^{l}(\mathscr{A}) \xrightarrow{\partial} D^{l-1}(\mathscr{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} D^{0}(\mathscr{A}) \to 0$$

which have the same modules $D^p(\mathscr{A})$ with different boundary maps. Several results on their homology groups will be obtained there. From the definition it is clear that $\Psi(\mathscr{A}; x, t)$ is a rational function. In Section 5, we use the results on the homology groups to prove that $\Psi$ is a polynomial in $x$ and $t$. Thus we can again rewrite the main Theorem (1.3) as

$$\chi(\mathscr{A}; t) = (-1)^l \Psi(\mathscr{A}; 1, t). \tag{1.4}$$

In Section 6, we characterize the characteristic polynomial $\chi(\mathscr{A}; t)$ by four conditions. We complete our proof of the main Theorem (1.4) be showing that $(-1)^l\, \Psi(\mathscr{A}; 1, t)$ satisfies the four conditions.

This paper does not contain any topological argument. It would be misleading though to avoid mention of the topological undercurrent in case $\mathbf{K} = \mathbf{C}$. In [6] Orlik and Solomon showed that the Poincaré polynomial of the topological space $M(\mathscr{A}) = \mathbf{C}^l \backslash \bigcup_{H \in \mathscr{A}} H$ is equal to $(-t)^l \chi(\mathscr{A}; -t^{-1})$. Thus, assuming the main Theorem (1.4), we have

(1.5) COROLLARY. *For any arrangement $\mathscr{A}$ in $C^l$, the Poincaré polynomial of $M(\mathscr{A})$ is $t^l \Psi(\mathscr{A}; 1, -t^{-1})$.*

We would like to thank J. Tits for asking N. Bourbaki about the origin of the substitution $y = t(x-1) - 1$ and for communicating his answer to us.

In closing, we note that results very similar in spirit to the ones proved in this paper had been conjectured in 1973 by Kelly and Rota [15].

## 2. THE MODULES $D^p(\mathscr{A})$

Let $\mathscr{A}$ be an arrangement in $V$. Define the polynomial $Q = Q_{\mathscr{A}} \in S$, the $S$-modules $\mathrm{Der}^p(S)$ and submodules $D^p(\mathscr{A})$ as in the Introduction. An element $\theta \in \mathrm{Der}^p(S)$ is called *homogeneous of degree* $d$, or $\deg \theta = d$, if $\theta(f_1, ..., f_p) \in S$ is zero or homogeneous of degree $d$ for any $f_1, ..., f_p \in V^*$. Then $\mathrm{Der}^p(S)$ is naturally a graded $S$-module. Note that an element $\theta \in \mathrm{Der}^p(S)$ is determined by its values on $V^* \times \cdots \times V^*$. Since $\theta$ is alternating linear, $\theta = 0$ if $p > l$. So $\mathrm{Der}^p(S) = 0$ for $p > l$. It is easy to see that $D^p(\mathscr{A})$ is a graded submodule of $\mathrm{Der}^p(S)$ for each $p$. Let $\mathbf{L}$ be a field extension of $\mathbf{K}$. Let $V_{\mathbf{L}} = \mathbf{L} \otimes_{\mathbf{K}} V$ and let $\mathscr{A}_{\mathbf{L}}$ be the corresponding arrangement in $V_{\mathbf{L}}$. Then $\mathbf{L} \otimes_{\mathbf{K}} D^p(\mathscr{A})$ is isomorphic to $D^p(\mathscr{A}_{\mathbf{L}})$. Thus the right-hand side of the main Theorem (1.2) is independent of field extension. Since

$\chi(\mathscr{A}_{\mathrm{L}}; t) = \chi(\mathscr{A}; t)$, we may assume, in the proof of the main Theorem, that $\mathbf{K}$ is algebraically closed.

(2.1) LEMMA. *If $\mathscr{A} \subseteq \mathscr{B}$ then $D^p(\mathscr{A}) \supseteq D^p(\mathscr{B})$ for any $p \geqslant 0$.*

*Proof.* Let $Q_1$ and $Q_1Q_2$ be defining equations for $\mathscr{A}$ and $\mathscr{B}$, respectively. For $\theta \in D^p(\mathscr{B})$ we have

$$\begin{aligned} Q_1\theta(Q_2, f_2, ..., f_p) &+ Q_2\theta(Q_1, f_2, ..., f_p) \\ &= \theta(Q_1Q_2, f_2, ..., f_p) \in Q_1Q_2S. \end{aligned}$$

Since $Q_1$ and $Q_2$ are coprime, we have $\theta(Q_1, f_2, ..., f_p) \in Q_1S$. ∎

Note that the graded $S$-module $\mathrm{Der}^p(S)$ is naturally isomorphic to the $p$th exterior power $\Lambda_S^p \mathrm{Der}(S)$ of $\mathrm{Der}(S) = \mathrm{Der}^1(S)$ over $S$. We identify the two. So there is a product structure $\mathrm{Der}^p(S) \times \mathrm{Der}^q(S) \to \mathrm{Der}^{p+q}(S)$. This makes $\bigoplus_{p=0}^{l} \mathrm{Der}^p(S)$ an $S$-algebra. Let $\varphi \wedge \psi \in \mathrm{Der}^{p+q}(S)$ denote the product of $\varphi \in \mathrm{Der}^p(S)$ and $\psi \in \mathrm{Der}^q(S)$. Explicitly [1, Chap. 3, Sect. 11, No. 2, Ex.3] we have

$$(\varphi \wedge \psi)(f_1, ..., f_{p+q}) = \sum \mathrm{sgn}(\sigma)\, \varphi(f_{\sigma 1}, ..., f_{\sigma p})\, \psi(f_{\sigma(p+1)}, ..., f_{\sigma(p+q)}) \qquad (2.2)$$

where the sum is over the set of all permutations $\sigma$ of $\{1, ..., p+q\}$ such that $\sigma 1 < \cdots < \sigma p$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. In particular for $\theta_1, ..., \theta_p \in \mathrm{Der}(S)$, we have

$$(\theta_1 \wedge \cdots \wedge \theta_p)(f_1, ..., f_p) = \det[\theta_i(f_j)]_{1 \leqslant i, j \leqslant p} \qquad (2.3)$$

for all $f_1, ..., f_p \in S$. By (2.2) we easily obtain

$$\text{If } \varphi \in D^p(\mathscr{A}) \text{ and } \psi \in D^q(\mathscr{A}), \text{ then } \varphi \wedge \psi \in D^{p+q}(\mathscr{A}). \qquad (2.4)$$

This implies that $\bigoplus_{p=0}^{l} D^p(\mathscr{A})$ also has an $S$-algebra structure.

Let $x_1, ..., x_l$ be a basis for $V^*$. Define $D_i \in \mathrm{Der}(S)$ to be the unique $\mathbf{K}$-derivation with $D_i(x_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant l)$. Then $D_1, ..., D_l$ are a basis for $\mathrm{Der}(S)$ over $S$. We sometimes write $\partial/\partial x_i$ instead of $D_i$.

(2.5) PROPOSITION. *The $S$-module $D^l(\mathscr{A})$ is free of rank one with a basis $Q(D_1 \wedge \cdots \wedge D_l)$.*

*Proof.* Let $\theta = f(D_1 \wedge \cdots \wedge D_l) \in D^l(\mathscr{A})$ with $f \in S$. Fix $H \in \mathscr{A}$. Let $\alpha \in V^*$ with $\ker(\alpha) = H$. Choose $i$ with $D_i(\alpha) \neq 0$. We can assume $i = 1$ and $D_1(\alpha) = 1$. Using (2.3) we compute $\theta(\alpha, x_2, ..., x_l) = f$. Because $\theta \in D^l(\mathscr{A}) \subseteq D^l(\{H\})$ by (2.1), $f \in \alpha S$. This is true for all $H \in \mathscr{A}$. Thus $f \in QS$. ∎

(2.6) DEFINITION. Let $M = \oplus_{p \geqslant 0} M_p$ be a finitely generated graded $S$-module. The *Poincaré series* $\operatorname{Poin}(M; x)$ of $M$ is defined by

$$\operatorname{Poin}(M; x) = \operatorname{Poin}(M) = \sum_{p \geqslant 0} \dim_{\mathbf{K}}(M_p)\, x^p.$$

It is known [5, Theorem 19, p. 346] that, for sufficiently large $p$, the function $\dim_{\mathbf{K}}(M_p)$ is a polynomial in $p$ of degree at most $l-1$. Thus $\operatorname{Poin}(M; x)$ is a rational function in $x$ with a pole only at $x = 1$ of order at most $l$. In other words $(x-1)^l \operatorname{Poin}(M; x)$ is a polynomial.

(2.7) DEFINITION. $\Psi(\mathscr{A}; x, t) = \sum_{p=0}^{l} \operatorname{Poin}(D^p(\mathscr{A}); x)\ (t(x-1)-1)^p$.

See the remark at the end of Section 3 for the source of this definition.

## 3. THE CASE OF A FREE ARRANGEMENT

In this section, we deal with free arrangements only. We compute $\Psi(\mathscr{A}; x, t)$ and obtain a formula for $\chi(\mathscr{A}; t)$ assuming the main Theorem. When $\mathbf{K} = \mathbf{C}$ this formula is the main result in [11].

(3.1) DEFINITION. We say that an arrangement $\mathscr{A}$ is *free* if $D^1(\mathscr{A})$ is a free $S$-module.

Assume that $\mathscr{A}$ is free. Then we can choose a homogeneous basis $\{\theta_1, ..., \theta_l\}$ for $D^1(\mathscr{A})$.

(3.2) DEFINITION. We call $\deg \theta_1, ..., \deg \theta_l$ the *exponents* of the free arrangement $\mathscr{A}$ and write $\exp \mathscr{A} = \{\deg \theta_1, ..., \deg \theta_l\}$.

The following criterion was proved by Saito [8] in the complex analytic category. This algebraic version was announced in [12]:

(3.3) PROPOSITION. *The elements $\theta_1, ..., \theta_l \in D^1(\mathscr{A})$ form a basis for $D^1(\mathscr{A})$ as an S-module if and only if $\theta_1 \wedge \cdots \wedge \theta_l \in \mathbf{K}^* Q(D_1 \wedge \cdots \wedge D_l)$* ($\mathbf{K}^* = \mathbf{K} \setminus \{0\}$).

*Proof* (if part). First note that $\theta_1, ..., \theta_l$ are linearly independent over $S$ because $\theta_1 \wedge \cdots \wedge \theta_l \neq 0$. We can assume $\theta_1 \wedge \cdots \wedge \theta_l = Q(D_1 \wedge \cdots \wedge D_l)$. Let $\theta \in D^1(\mathscr{A})$. By Cramer's rule $QD_i \in S\theta_1 + \cdots + S\theta_l$ $(i = 1, ..., l)$. So write $Q\theta = f_1\theta_1 + \cdots + f_l\theta_l$. Then we have

$$\begin{aligned} f_i Q(D_1 \wedge \cdots \wedge D_l) &= \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge (Q\theta) \wedge \theta_{i+1} \wedge \cdots \wedge \theta_l \\ &\in Q\, D^l(\mathscr{A}) = SQ^2(D_1 \wedge \cdots \wedge D_l) \end{aligned}$$

by (2.5). Thus $f_i \in QS$ for each $i$. So $\theta = (f_1/Q)\,\theta_1 + \cdots + (f_l/Q)\,\theta_l \in S\theta_1 + \cdots + S\theta_l$.

(only if part). By (2.4) and (2.5) we can write $\theta_1 \wedge \cdots \wedge \theta_l = fQ(D_1 \wedge \cdots \wedge D_l)$ for some $f \in S$. Fix $H \in \mathscr{A}$. We can assume that $x_1$ is a defining equation of $H$. Then $Q_H = Q/x_1$ is a defining equation of $\mathscr{A} \setminus \{H\}$. Since $QD_1$ and $Q_H D_i$ $(i = 2,\dots, l)$ are in $D^1(\mathscr{A})$, we have

$$\begin{aligned} Q\,Q_H^{l-1}(D_1 \wedge \cdots \wedge D_l) &= (x_1 Q_H D_1) \wedge (Q_H D_2) \wedge \cdots \wedge (Q_H D_l) \\ &\in S(\theta_1 \wedge \cdots \wedge \theta_l) = SfQ(D_1 \wedge \cdots \wedge D_l). \end{aligned}$$

Therefore $f$ divides $Q_H^{l-1}$. This is true for all $H \in \mathscr{A}$. Since there are no common factors of $\{Q_H^{l-1}\}_{H \in \mathscr{A}}$, we have $f \in \mathbf{K}^*$. ∎

(3.4) PROPOSITION. *Let $\mathscr{A}$ be a free arrangement and $\{\theta_1,\dots, \theta_l\}$ be a homogeneous basis for $D^1(\mathscr{A})$. Then for any $p$, $1 \leqslant p \leqslant l$, $D^p(\mathscr{A})$ is a free $S$-module with a basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_p} \mid 1 \leqslant i_1 < \cdots < i_p \leqslant l\}$.*

*Proof.* Let $\theta \in D^p(\mathscr{A})$. Let $I = (i_1,\dots, i_p)$, where $1 \leqslant i_1 < \cdots < i_p \leqslant l$. Write $D_I = D_{i_1} \wedge \cdots \wedge D_{i_p} \in \operatorname{Der}^p(S)$. Similarly $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$. Since $QD_i \in D^1(\mathscr{A}) = S\theta_1 + \cdots + S\theta_l$, $Q^p \operatorname{Der}^p(S) \subseteq \sum_I S\theta_I$. Write $Q^p\theta = \sum_I f_I \theta_I$ for $f_I \in S$. Choose $J \subseteq \{1,\dots, l\}$ of size $l-p$. Then, by (3.3),

$$Q^p(\theta \wedge \theta_J) = \left(\sum_I f_I \theta_I\right) \wedge \theta_J = f_K Q(D_1 \wedge \cdots \wedge D_l), \tag{3.5}$$

where $K$ is the complement of $J$. By (2.4) and (2.5) we have $\theta \wedge \theta_J \in D^l(\mathscr{A}) = SQ(D_1 \wedge \cdots \wedge D_l)$. This implies that $Q^p$ divides $f_K$. This is true for all $K$. So $\theta = \sum_I (f_I/Q^p)\,\theta_I \in \sum_I S\theta_I$. Thus the elements $\theta_I$ span $D^p(\mathscr{A})$. If $\sum_I f_I \theta_I = 0$, then by the second equality in (3.5) we have $f_I = 0$. ∎

*Remark.* Combining (3.3) and (3.4) we know the following four conditions are equivalent:

(1) $\mathscr{A}$ is free,

(2) $\Lambda_S^l D^1(\mathscr{A}) = D^l(\mathscr{A})$,

(3) $\Lambda_S^p D^1(\mathscr{A}) = D^p(\mathscr{A})$ for $p = 0,\dots, l$,

(4) each $D^p(\mathscr{A})$ is free for $p = 0,\dots, l$.

(3.6) COROLLARY. *Let $\mathscr{A}$ be a free arrangement with* $\exp \mathscr{A} = \{b_1,\dots, b_l\}$. *Then*

$$\Psi(\mathscr{A}; x, t) = \prod_{i=1}^{l} (1 + x + x^2 + \cdots + x^{b_i - 1} - x^{b_i} t).$$

*In particular* $\Psi(\mathscr{A}; 1, t) = \prod_{i=1}^{l} (b_i - t)$.

*Proof.* Since $\deg \theta_{i_1} \wedge \cdots \wedge \theta_{i_p} = b_{i_1} + \cdots + b_{i_p}$, (3.4) shows that

$$\mathrm{Poin}(D^p(\mathscr{A}); x) = \sum x^{b_{i_1} + \cdots + b_{i_p}}/(1-x)^l,$$

where the sum is over the set $\{(i_1,..., i_p) | 1 \leqslant i_1 < \cdots < i_p \leqslant l\}$. Let $y$ be an indeterminate. Then

$$\sum_{p=0}^{l} \mathrm{Poin}(D^p(\mathscr{A}); x)\, y^p = \prod_{i=1}^{l} ((1 + x^{b_i} y)/(1-x)). \tag{3.7}$$

Replace $y$ by $t(x-1)-1$. An easy computation, together with the definition (2.7) of $\Psi(\mathscr{A}; x, t)$, proves the formula. ∎

(3.8) COROLLARY. *Let $\Phi_l$ denote the empty arrangement in an $l$-dimensional vector space. Then $\Psi(\Phi_l; x, t) = (-t)^l$.*

*Proof.* This follows from $\exp(\Phi_l) = \{0,..., 0\}$. ∎

Assuming the main Theorem (1.4), we have

(3.9 COROLLARY. *Let $\mathscr{A}$ be a free arrangement with* $\exp \mathscr{A} = \{b_1,..., b_l\}$. *Then*

$$\chi(\mathscr{A}; t) = \prod_{i=1}^{l} (t - b_i). \quad ∎$$

When $\mathbf{K} = \mathbf{C}$, this is the factorization theorem for free arrangements in [11]. For a general field $\mathbf{K}$, this has been announced in [12].

*Remark.* Bourbaki [2, Chap. 5, Sect. 5, Exer. 3] introduced the substitution $t = (y+1)/(x-1)$ in a problem on unitary reflection groups, as a device to convert information about a bigraded module into a polynomial identity in one variable. The same substitution was used by Orlik and Solomon in a similar situation in [7]. Since the main results on reflection groups in [7] are implicit in our main theorem here, it seems appropriate to clarify the connection between [7] and this paper. Suppose $\mathbf{K} = \mathbf{C}$. Let $G \subseteq GL(V)$ be a finite unitary reflection group and let $R = S^G$ be the ring of $G$-invariants. We know from Chevalley's theorem that $R$ may be generated by $l$ homogeneous elements of degrees, say $d_1,..., d_l$. It follows from [7, 2.3] with $M = V^*$ that $\mathrm{Der}(S)^G$ is a free $R$-module with a homogeneous basis, say $\theta_1,..., \theta_l$. Since there is a natural isomorphism $\mathrm{Der}^p(S) \cong S \otimes_{\mathbf{C}} \Lambda^p(V)$ of graded $\mathbf{C}[G]$-modules, Theorem 3.1 of [7] shows that $\mathrm{Der}^p(S)^G$ is a free $R$-module with basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_p} | 1 \leqslant i_1 < \cdots < i_p \leqslant l\}$. Define integers $b_i = \deg \theta_i$ for $1 \leqslant i \leqslant l$. It follows from [7, 3.5] that

$$\sum_{p=0}^{l} \mathrm{Poin}(\mathrm{Der}^p(S)^G; x) = \prod_{i=1}^{l} (1 + x^{b_i} y)/(1 - x^{d_i}). \tag{3.10}$$

It follows from (3.10) using Molien's formula and the substitution $t=(y+1)/(x-1)$ that [7, Corollary 3.10]

$$\sum_{g\in G} \det(g)\, t^{k(g)} = \prod_{i=1}^{l} (t-b_i), \tag{3.11}$$

where $k(g)$ is the dimension of the fixed point set of $g$. Now the argument in Section 4 of [7] shows that if $\mathscr{A}=\mathscr{A}(G)$ is the arrangement of all reflecting hyperplanes of $G$ then

$$\chi(\mathscr{A}; t) = \prod_{i=1}^{l} (t-b_i). \tag{3.12}$$

It is shown in [10, Theorem 2 and Proposition 2] using Saito's criterion (3.3) that $\mathscr{A}=\mathscr{A}(G)$ is a free arrangement and that $\{\theta_1,..., \theta_l\}$ is a basis for $D^1(\mathscr{A})$ as an $S$-module. It follows from (3.4) that $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_p} \mid 1 \leqslant i_1 < \cdots < i_p \leqslant l\}$ is a basis for $D^p(\mathscr{A})$ as an $S$-module. Therefore we have

$$D^p(\mathscr{A}) \cong \operatorname{Der}^p(S)^G \otimes_R S.$$

Since there exists a graded subspace $U$ of $S$ with $S \cong R \otimes_{\mathbf{C}} U$ [2, Chap. 5, No. 3, Theorem 2], the Poincaré series of $D^p(\mathscr{A})$ as an $S$-module is equal to $\prod_{i=1}^{l} ((1-x^{d_i})/(1-x))$ times the Poincaré series of $\operatorname{Der}^p(S)^G$ as an $R$-module. Thus, if $\mathscr{A}=\mathscr{A}(G)$, we know that (3.10) and (3.7) are equivalent and that the argument in this paper allows us to bypass (3.11) and prove (3.12) directly.

## 4. The Complexes $D^{\cdot}(\mathscr{A})$

In this section we construct a family of complexes which have the same modules $D^p(\mathscr{A})$ $(p \geqslant 0)$ with different boundary maps. Let $h \in S$. Define an $S$-linear map

$$i(h)\colon \operatorname{Der}^p(S) \to \operatorname{Der}^{p-1}(S)$$

by

$$(i(h)(\theta))(f_2,..., f_p) = \theta(h, f_2,..., f_p)$$

for any $f_2,..., f_p \in S$. This map is known as the *interior product* when we regard it as a map from $\Lambda_S^p \operatorname{Der}(S)$ to $\Lambda_S^{p-1} \operatorname{Der}(S)$.

(4.1) Lemma [1, Chap. 3, Sect. 11, No. 8].

(i) $i(h)(\varphi \wedge \psi) = (i(h)\,\varphi) \wedge \psi + (-1)^p\, \varphi \wedge (i(h)\,\psi)$ *for any* $\varphi \in \mathrm{Der}^p(S)$ *and* $\psi \in \mathrm{Der}^q(S)$,

(ii) $i(h) \cdot i(h) = 0$. ∎

Assume that $\mathscr{A}$ is non-empty. Fix a hyperplane $H \in \mathscr{A}$. Let $\alpha = \alpha_H$ denote its defining equation. Note, as in the proof of (2.5), that if $\theta \in D^p(\mathscr{A})$ and $f_2, ..., f_p \in S$, then $\theta(\alpha, f_2, ..., f_p) \in \alpha S$. Thus we may define $\partial\colon D^p(\mathscr{A}) \to D^{p-1}(\mathscr{A})$ by

$$(\partial\theta)(f_2, ..., f_p) = (i(\alpha)\,\theta)(f_2, ..., f_p)/\alpha = \theta(\alpha, f_2, ..., f_p)/\alpha.$$

This map $\partial$ is homogeneous of degree $-1$. Since $\partial \cdot \partial = 0$ by (4.1.ii), we have a complex

$$0 \longrightarrow D^l(\mathscr{A}) \xrightarrow{\partial} D^{l-1}(\mathscr{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} D^0(\mathscr{A}) \to 0 \tag{4.2}$$

unless $\mathscr{A}$ is the empty arrangement. By (4.1.i) we have

$$\partial(\varphi \wedge \psi) = (\partial\varphi) \wedge \psi + (-1)^p \varphi \wedge (\partial\psi) \text{ for any } \varphi \in D^p(\mathscr{A}) \text{ and } \psi \in D^q(\mathscr{A}). \tag{4.3}$$

(4.4) Proposition. *The complex* (4.2) *is acyclic.*

*Proof.* Let $\theta_E$ denote the Euler derivation: $\theta_E = \sum_{i=1}^{l} x_i D_i$. Then $\theta_E \in D^1(\mathscr{A})$. Let $\varphi \in D^p(\mathscr{A})$ be a cycle: $\partial\varphi = 0$. By (2.4), $\theta_E \wedge \varphi \in D^{p+1}(\mathscr{A})$. By (4.3) we have

$$\partial(\theta_E \wedge \varphi) = (\partial\theta_E) \wedge \varphi = ((\theta_E(\alpha))/\alpha)\,\varphi = \varphi. \quad ∎$$

We shall define other boundary maps on $\bigoplus_{p \geq 0} D^p(\mathscr{A})$. Let $X$ be a subspace of $V$. Let $S^X = S(X^*)$ be the symmetric algebra of the dual space $X^*$ of $X$. For a subset $I$ of $S^X$, let $V(I)$ be the set of common zeros in $X$ of all elements of $I$. We say that $f \in S^X$ is *non-degenerate* on $X$ if $V(J(f)) \subseteq \{0\}$, where $J(f) = \{\theta(f) \in S^X \mid \theta \in \mathrm{Der}(S^X)\}$ is the Jacobian ideal of $f$. If $z_1, ..., z_k$ are a basis for $X^*$, we can rewrite this condition as $V(\partial f/\partial z_1, ..., \partial f/\partial z_k) \subseteq \{0\}$. Let $S_d$ denote the set of all homogeneous elements of degree $d$ of $S$.

(4.5) Definition. *If $X$ is a subspace of $V$, let $N_d^X$ be the set of all $h \in S_d$ such that the restriction $h|_X \in S^X$ is non-degenerate on $X$.*

Let $x_1, ..., x_l$ be a basis for $V$ such that $X = V(x_{k+1}, ..., x_l)$. Note that $h \in S_d$ belongs to $N_d^X$ if and only if $V(\partial h/\partial x_1, ..., \partial h/\partial x_k) \cap X \subseteq \{0\}$. Let $d$ be a positive integer which is not a multiple of the characteristic of the field $\mathbf{K}$. When $\dim X = 1$, we easily see that $N_d^X$ is Zariski dense in the space $S_d$. Let

$l \geqslant \dim X \geqslant 2$. Projectivize $V$ and $X$ to get $\mathbf{P}(V)$ and $\mathbf{P}(X)$. Then $\mathbf{P}(X)$ is naturally a linear subspace of the $(l-1)$-dimensional projective space $\mathbf{P}(V)$. Let $Y$ be the hypersurface of degree $d$ in $\mathbf{P}(V)$ defined by $h$. Then $h \in N_d^X$ if and only if the intersection $\mathbf{P}(X) \cap Y \neq \mathbf{P}(X)$ is non-singular. Since $\mathbf{K}$ is algebraically closed, it follows from Bertini's theorem [3, Theorem 8.18, Example 8.20.2] that $N_d^X$ is Zariski dense in the space $S_d$. (This also follows from the existence of the resultant in elimination theory [13, Vol. II, Sect. 80, p. 8].) Since $L(\mathscr{A})$ is finite,

$$\bigcap_{\substack{X \in L(\mathscr{A}) \\ \dim X > 0}} N_d^X$$

is Zariski dense, and in particular is nonempty. Choose

$$h = h_d \in \bigcap_{\substack{X \in L(\mathscr{A}) \\ \dim X > 0}} N_d^X. \tag{4.6}$$

Define $\partial: D^p(\mathscr{A}) \to D^{p-1}(\mathscr{A})$ by $\partial\theta = i(h)(\theta)$ for $\theta \in D^p(\mathscr{A})$. This map $\partial$ is homogeneous of degree $d-1$. It is $S$-linear because $i(h)$ is. We have a complex

$$0 \longrightarrow D^l(\mathscr{A}) \xrightarrow{\partial} D^{l-1}(\mathscr{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} D^0(\mathscr{A}) \to 0 \tag{4.7}$$

in which the boundary operator depends on $h$.

By (4.1.i) we have

$$\partial(\varphi \wedge \psi) = (\partial\varphi) \wedge \psi + (-1)^p \varphi \wedge (\partial\psi) \text{ for any } \varphi \in D^p(\mathscr{A}) \text{ and } \psi \in D^q(\mathscr{A}). \tag{4.8}$$

Let $S_+$ be the maximal ideal of $S = S(V^*)$ generated by all homogeneous elements of positive degree.

(4.9) LEMMA. *The radical of the ideal* $D^1(\mathscr{A})\,h = \{\theta(h) \in S \mid \theta \in D^1(\mathscr{A})\}$ *contains the maximal ideal* $S_+$.

*Proof.* It is enough, by Hilbert's Nullstellensatz, to show that the zero locus $V(D^1(\mathscr{A})\,h)$ is contained in $\{0\}$. Let $v \in V \setminus \{0\}$. Choose the minimal $X \in L(\mathscr{A})$ containing $v$. Thus $v \in X$ and $v \notin Y$ for any $Y \in L(\mathscr{A})$ with $Y \subset X$. We can take a basis $x_1, ..., x_l$ for $V^*$ such that $X = V(x_{k+1}, ..., x_l)$. Let $\mathscr{A}_1 = \{H \in \mathscr{A} \mid H \not\supseteq X\}$ and let $Q_1$ be its defining equation. Note that $Q_1(\partial/\partial x_i) \in D^1(\mathscr{A})$ $(i = 1, ..., k)$. Since $h \in N_d^X$, we have $V(\partial h/\partial x_1, ..., \partial h/\partial x_k) \cap X \subseteq \{0\}$. In particular $v \notin V(\partial h/\partial x_1, ..., \partial h/\partial x_k)$. Since $X$ is minimal, we have $Q_1(v) \neq 0$. Thus

$$v \notin V(Q_1(\partial h/\partial x_1), ..., Q_1(\partial h/\partial x_k)) \supseteq V(D^1(\mathscr{A})\,h).$$

This implies $v \notin V(D^1(\mathscr{A})\,h)$. ∎

(4.10) PROPOSITION. *The homology groups of the complex* (4.7) *are finite dimensional over* $\mathbf{K}$.

*Proof.* Let $H_p$ be the $p$th homology of the complex (4.7). Note that $H_p$ is an $S$-module. Since $D^p(\mathscr{A})$ is a finitely generated $S$-module, so is $H_p$. Thus it is sufficient, thanks to (4.9), to prove that $D^1(\mathscr{A})\,h$ annihilates $H_p$. Let $\theta \in D^1(\mathscr{A})$. Take a cycle $\varphi \in D^p(\mathscr{A})$. We have $\theta(h)\,\varphi = (\partial\theta)\,\varphi = \partial(\theta \wedge \varphi)$ by (4.8). Thus $\theta(h)\,\varphi$ is a boundary because $\theta \wedge \varphi \in D^{p+1}(\mathscr{A})$ by (2.4). ∎

*Remark.* We have a few pieces of evidence suggesting that the homology $H_p$ $(p>0)$ of the complex (4.7) always vanishes: (i) $H_l = H_{l-1} = 0$ for all arrangements, (ii) $H_p = 0$ $(p>0)$ for free arrangements and (iii) $H_p = 0$ $(p>0)$ for arrangements in $\mathbf{R}^3$.

## 5. SOME PROPOERTIES OF $\Psi(\mathscr{A}; x, t)$

(5.1) DEFINITION. Define the Poincaré series

$$\operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, y) = \sum_{p \geqslant 0} \operatorname{Poin}(D^p(\mathscr{A}); x)\, y^p$$

of $D^{\cdot}(\mathscr{A}) = \bigoplus_{p \geqslant 0} D^p(\mathscr{A})$.

Since each $D^p(\mathscr{A})$ is a finitely generated graded $S$-module, $\operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, y)$ is a rational function in $x$ with a pole only at $x=1$ of order at most $l$. It is a polynomial in $y$ of degree $l$. Thus $(x-1)^l \operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, y)$ is a polynomial in $x$ and $y$.

(5.2) PROPOSITION. *Let $d$ be a non-negative integer such that $d+1$ is not a multiple of the characteristic of the field* $\mathbf{K}$. *Then* $\operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, -x^d)$ *is a polynomial in $x$.*

*Proof.* Choose

$$h = h_{d+1} \in \bigcap_{\substack{X \in L(\mathscr{A}) \\ \dim X > 0}} N^X_{d+1} \subseteq S_{d+1}$$

as in (4.6). We use the complex

$$0 \longrightarrow D^l(\mathscr{A}) \xrightarrow{\partial} D^{l-1}(\mathscr{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} D^0(\mathscr{A}) \longrightarrow 0 \qquad (4.7)$$

in which each boundary map $\partial = i(h)$ is of degree $d$. Since the homology groups of the complex are finite-dimensional by (4.10),

$$\begin{aligned}\operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, -x^d) &= \sum_{p=0}^{l} \operatorname{Poin}(D^p(\mathscr{A}); x)(-x^d)^p \\ &= \sum_{p=0}^{l} \operatorname{Poin}(H_p(D^{\cdot}(\mathscr{A})); x)(-x^d)^p\end{aligned}$$

is a polynomial. ∎

Recall the definition $\Psi(\mathscr{A}; x, t) = \operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, t(x-1)-1)$ in (2.7).

(5.3) PROPOSITION. *$\Psi(\mathscr{A}; x, t)$ is a polynomial in $x$ and $t$.*

*Proof.* Since $(x-1)^l \operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, y)$ is a polynomial, we can expand it as

$$\sum_{i,j \geqslant 0} c_{ij}(x-1)^i(y+1)^j$$

with $c_{ij} \in \mathbf{Q}$. Let $d \geqslant 0$. Then we have, by setting $t = (1-x^d)/(x-1)$,

$$\begin{aligned}&(x-1)^l\, \Psi(\mathscr{A}; x, (1-x^d)/(x-1)) \\ &\quad = (x-1)^l \operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, -x^d) \\ &\quad = \sum_{i,j \geqslant 0} c_{ij}(x-1)^{i+j}(-1-x-\cdots-x^{d-1})^j \\ &\quad = \sum_{k \geqslant 0}\left(\sum_{i+j=k} c_{ij}(-1-x-\cdots-x^{d-1})^j\right)(x-1)^k.\end{aligned}$$

By applying (5.2), we know that $(x-1)^l$ divides this polynomial for infinitely many non-negative integers $d$. Thus we get a family of systems of equations

$$\begin{aligned}&c_{00} = 0 \\ &c_{10} - d\, c_{01} = 0 \\ &c_{20} - d\, c_{11} + d^2\, c_{02} = 0 \\ &\quad\vdots \\ &c_{l-1,0} - d\, c_{l-2,1} + \cdots + (-d)^{l-1}\, c_{0,l-1} = 0\end{aligned}$$

parametrized by infinitely many $d$. These have only one solution $c_{ij} = 0$ $(0 \leqslant i+j < l)$ because the Vandermonde determinant is not zero. This implies that $(x-1)^l$ divides the polynomial

$$(x-1)^l\, \Psi(\mathscr{A}; x, t) = \sum_{i,j \geqslant 0} c_{ij}(x-1)^{i+j} t^j. \quad ∎$$

(5.4) PROPOSITION. *If $\mathscr{A}$ is not the empty arrangement, then* $\operatorname{Poin}(D^{\cdot}(\mathscr{A}); x, -x^{-1}) = 0$.

*Proof.* Since $\mathscr{A}$ is nonempty, we may use the complex

$$0 \longrightarrow D^l(\mathscr{A}) \xrightarrow{\partial} D^{l-1}(\mathscr{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} D^0(\mathscr{A}) \longrightarrow 0 \tag{4.2}$$

in which each boundary map $\partial$ is of degree $-1$. Since the complex is acyclic by (4.4), we obtain

$$\begin{aligned} x^l \operatorname{Poin}(D^\cdot(\mathscr{A}); x, -x^{-1}) &= \sum_{p=0}^{l} \operatorname{Poin}(D^p(\mathscr{A}); x)(-x)^{l-p} \\ &= \sum_{p=0}^{l} \operatorname{Poin}(H_p(D^\cdot(\mathscr{A})); x)(-x)^{l-p} = 0. \quad \blacksquare \end{aligned}$$

(5.5) PROPOSITION. *If $\mathscr{A}$ is not the empty arrangement, then* $\Psi(\mathscr{A}; x, x^{-1}) = 0$.

*Proof.* By (5.4), we have

$$\begin{aligned} \Psi(\mathscr{A}; x, x^{-1}) &= \operatorname{Poin}(D^\cdot(\mathscr{A}); x, x^{-1}(x-1)-1) \\ &= \operatorname{Poin}(D^\cdot(\mathscr{A}); x, -x^{-1}) = 0. \quad \blacksquare \end{aligned}$$

(5.6) DEFINITION. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be arrangements in vector spaces $V_1$ and $V_2$, respectively. Define the *product*

$$\mathscr{A}_1 \times \mathscr{A}_2 = \{H \oplus V_2 \mid H \in \mathscr{A}_1\} \cup \{V_1 \oplus K \mid K \in \mathscr{A}_2\}.$$

This is an arrangement in $V = V_1 \oplus V_2$. Put $S = S(V^*)$, $S_1 = S(V_1^*)$, and $S_2 = S(V_2^*)$. In the rest of the Section tensor product $\otimes$ is always over $\mathbf{K}$. For $i = 1, 2$, we view $S_i \subseteq S$ and $\operatorname{Der}^1(S_i) \subseteq \operatorname{Der}^1(S)$ in the natural way so that $\operatorname{Der}^1(S_1)$ annihilates $S_2$. Then we may identify $S$ with $S_1 \otimes S_2$ and $\operatorname{Der}^n(S)$ with $\bigoplus_{p+q=n} \operatorname{Der}^p(S_1) \wedge \operatorname{Der}^q(S_2) \cong \bigoplus_{p+q=n} \operatorname{Der}^p(S_1) \otimes \operatorname{Der}^q(S_2)$ as graded vector spaces. Let $Q_i \in S_i$ be a defining equation for $\mathscr{A}_i$ and let $\Phi_i$ be the empty arrangement in $V_i$ $(i = 1, 2)$.

(5.7) LEMMA. *Let $\varphi_1, ..., \varphi_m \in \operatorname{Der}^p(S_1)$ and $\psi_1, ..., \psi_m \in \operatorname{Der}^q(S_2)$. Let $n = p + q$. If $\psi_1, ..., \psi_m$ are linearly independent over* $\mathbf{K}$ *and $\sum_{i=1}^m \varphi_i \otimes \psi_i \in D^n(\mathscr{A}_1 \times \Phi_2)$, then each $\varphi_i$ lies in $D^p(\mathscr{A}_1)$.*

*Proof.* Fix $f_2, ..., f_p \in S_1$. Define

$$\psi = \sum_{i=0}^{m} \varphi_i(Q_1, f_2, ..., f_p) \otimes \psi_i \in S_1 \otimes \operatorname{Der}^q(S_2) \subseteq \operatorname{Der}^q(S).$$

For any $g_1, ..., g_q \in S_2$, we have

$$\psi(g_1, ..., g_q) = \left( \sum_{i=1}^{m} \varphi_i \otimes \psi_i \right) (Q_1, f_2, ..., f_p, g_1, ..., g_q) \in Q_1 S.$$

This implies $\psi \in Q_1 S_1 \otimes \mathrm{Der}^q(S_2)$. Since $\psi_1, \dots, \psi_m$ are linearly independent over $\mathbf{K}$, $1 \otimes \psi_1, \dots, 1 \otimes \psi_m \in S_1 \otimes \mathrm{Der}^q(S_2)$ are independent over $S_1$. Thus each $\varphi_i(Q_1, f_2, \dots, f_p)$ lies in $Q_1 S_1$. ∎

(5.8) PROPOSITION. $D^n(\mathscr{A}_1 \times \mathscr{A}_2) \cong \bigoplus_{p+q=n} D^p(\mathscr{A}_1) \otimes_{\mathbf{K}} D^q(\mathscr{A}_2)$.

*Proof.* Let $\theta \in D^n(\mathscr{A}_1 \times \mathscr{A}_2)$. To show $\theta \in \bigoplus_{p+q=n} D^p(\mathscr{A}_1) \otimes D^q(\mathscr{A}_2)$ we can assume $\theta \in \mathrm{Der}^p(S_1) \otimes \mathrm{Der}^q(S_2)$. Write $\theta = \sum_{i=1}^m \varphi_i \otimes \psi_i$ for $\varphi_1, \dots, \varphi_m \in \mathrm{Der}^p(S_1)$ and $\psi_1, \dots, \psi_m \in \mathrm{Der}^q(S_2)$, where $\psi_1, \dots, \psi_m$ are linearly independent over $\mathbf{K}$. Since $\theta \in D^n(\mathscr{A}_1 \times \mathscr{A}_2) \subseteq D^n(\mathscr{A}_1 \times \Phi_2)$, each $\varphi_i$ lies in $D^p(\mathscr{A}_1)$ by (5.7). Thus $\theta \in D^p(\mathscr{A}_1) \otimes \mathrm{Der}^q(S_2)$. Write $\theta = \sum_{i=1}^n \xi_i \otimes \eta_i$ for $\xi_1, \dots, \xi_n \in D^p(\mathscr{A}_1)$ and $\eta_1, \dots, \eta_n \in \mathrm{Der}^q(S_2)$, where $\xi_1, \dots, \xi_n$ are linearly independent over $\mathbf{K}$. Since $\theta \in D^n(\mathscr{A}_1 \times \mathscr{A}_2) \subseteq D^n(\Phi_1 \times \mathscr{A}_2)$, each $\eta_i$ lies in $D^q(\mathscr{A}_2)$ by (5.7) again. Thus $\theta \in D^p(\mathscr{A}_1) \otimes D^q(\mathscr{A}_2)$. ∎

(5.9) DEFINITION. For $X \in L(\mathscr{A})$ define $\mathscr{A}_X = \{H \in \mathscr{A} \mid X \subseteq H\}$.

(5.10) PROPOSITION. *For any $X \in L(\mathscr{A})$, $t^{\dim X}$ divides the polynomial $\Psi(\mathscr{A}_X; x, t)$.*

*Proof.* Let $d = \dim X$. Note that $\mathscr{A}_X = \mathscr{A}_1 \times \Phi_d$ for some arrangement $\mathscr{A}_1$ and $\Phi_d$ the empty arrangement in $X$. By (5.8), we have

$$\mathrm{Poin}(D^{\cdot}(\mathscr{A}_1 \times \Phi_d); x, y) = \mathrm{Poin}(D^{\cdot}(\mathscr{A}_1); x, y)\, \mathrm{Poin}(D^{\cdot}(\Phi_d); x, y).$$

Let $y = t(x-1) - 1$ here to get $\Psi(\mathscr{A}_1 \times \Phi_d; x, t) = \Psi(\mathscr{A}_1; x, t)\, \Psi(\Phi_d; x, t)$. Since $\Psi(\Phi_d; x, t) = (-t)^d$ by (3.8), the result follows. ∎

## 6. PROOF OF THE MAIN THEOREM

Fix an arrangement $\mathscr{A}$ in $V$. Let $L(\mathscr{A})$ be the collection of all intersections of elements of $\mathscr{A}$. We agree that $\bigcap_{H \in \phi} H = V \in L(\mathscr{A})$. Introduce a partial order $\leqslant$ on $L(\mathscr{A})$ by $X \leqslant Y \Leftrightarrow X \supseteq Y$. Then $L(\mathscr{A})$ is a geometric lattice and

$$X \wedge Y = \bigcap \{H \in \mathscr{A} \mid H \supseteq X \cup Y\}, \qquad X \vee Y = X \cap Y$$

for all $X, Y \in \mathscr{A}$. Write $L = L(\mathscr{A})$. Let $\mu: L \times L \to \boldsymbol{Z}$ be the Möbius function [9, p. 116], which is characterized by

$$\sum_{\substack{Y \in L \\ W \leqslant Y \leqslant X}} \mu(Y, X) = \begin{cases} 1 & \text{if } W = X, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Let $L_X = \{Z \in L \mid Z \leqslant X\}$ for $X \in L$. Then $L_X = L(\mathscr{A}_X)$ by (5.9). Thus by the definition (1.1) of the characteristic polynomial $\chi$ we have $\chi(\mathscr{A}_X; t) = \sum_{Y \in L_X} \mu(V, Y)\, t^{\dim X}$. It follows from the property (6.1) of the Möbius function that the function on $L$ defined by $X \mapsto \chi(\mathscr{A}_X; t)$ has the properties (1)–(4) of the following proposition, which characterizes $\chi$ as in [11, Proposition 6.7]:

(6.2) PROPOSITION. *Suppose that a map $G\colon L \to \mathbf{Q}[t]$ satisfies the following four conditions:*

(1) $G(V) = t^l$,

(2) $G(X)|_{t=1} = 0$ *unless* $X = V$,

(3) $t^{\dim X}$ *divides* $G(X)$ *for any* $X \in L$,

(4) *the degree in $t$ of* $\sum_{Y \in L_X} \mu(Y, X)\, G(Y)$ *does not exceed*

$\dim X$ *for any* $X \in L$.

*Then $G(X) = \chi(\mathscr{A}_X; t)$ for any $X \in L$.*

*Proof.* Put $G'(X) = \sum_{Y \in L_X} \mu(Y, X)\, G(Y)$. If $Y \leqslant X$, then $\dim Y \geqslant \dim X$. By (3) $t^{\dim Y}$ divides $G(Y)$, so $t^{\dim X}$ divides $G'(X)$. On the other hand, by (4), $\deg G'(X) \leqslant \dim X$. Therefore one can write $G'(X) = g(X)\, t^{\dim X}$ for some map $g\colon L \to \mathbf{Q}$. We get

$$g(X) = G'(X)|_{t=1} = \sum_{Y \in L_X} \mu(Y, X)\, G(Y)|_{t=1} = \mu(V, X)$$

by using (1) and (2). Thus $G'(X) = \mu(V, X)\, t^{\dim X}$. By (6.1), we have

$$G(X) = \sum_{Y \in L_X} G'(Y) = \sum_{Y \in L_X} \mu(V, Y)\, t^{\dim Y} = \chi(\mathscr{A}_X; t). \quad \blacksquare$$

Define a map $G\colon L \to \mathbf{Z}[t]$ by

$$G(X) = (-1)^l\, \Psi(\mathscr{A}_X; 1, t). \tag{6.3}$$

In order to prove the main Theorem (1.4) it is sufficient to verify the conditions (1)–(4) in (6.2) for this $G$. We have already verified the first three conditions:

(1) $G(V) = (-1)^l\, \Psi(\Phi_l; 1, t) = t^l$ by (3.8),

(2) if $X \neq V$, then $G(X)|_{t=1} = (-1)^l\, \Psi(\mathscr{A}_X; 1, 1) = 0$ by (5.5) with $x = 1$,

(3) follows from (5.10) with $x = 1$.

*Thus the only remaining condition to be verified to complete our proof is* (4).

Regard the lattice $L$ as a category with morphisms $\leqslant$:

$$\mathrm{Hom}(X, Y) = \begin{cases} \{X \leqslant Y\} & \text{if } X \leqslant Y, \\ \varnothing & \text{otherwise.} \end{cases}$$

The composition of two morphisms $X \leqslant Y$ and $Y \leqslant Z$ is $X \leqslant Z$.

Let $\mathscr{C}$ be the category of finitely generated graded $S$-modules with $S$-linear maps which are homogeneous of degree zero as morphisms. Let $F$ be a contravariant functor from $L$ to $\mathscr{C}$. When $X, Y \in L$ with $X \geqslant Y$, the induced $S$-linear map homogeneous of degree zero from $F(X)$ to $F(Y)$ is denoted by $\nu_{X,Y}$ or simply by $\nu$. For $\mathfrak{p} \in \operatorname{Spec} S$ define $T(\mathfrak{p}) \in L$ by

$$T(\mathfrak{p}) = \bigcap \{H \in \mathscr{A} \mid \text{the defining equation of } H \text{ lies in } \mathfrak{p}\}.$$

(6.4) Definition. A contravariant functor $F\colon L \to \mathscr{C}$ is said to be *local* if the localization of the $S$-linear map

$$\nu\colon F(X) \to F(X \wedge T(\mathfrak{p}))$$

at $\mathfrak{p}$ is an isomorphism for any $\mathfrak{p} \in \operatorname{Spec} S$;

$$\nu_{\mathfrak{p}}\colon F(X)_{\mathfrak{p}} \cong F(X \wedge T(\mathfrak{p}))_{\mathfrak{p}}.$$

Fix $p \geqslant 0$. Consider the correspondence $X \to D^p(\mathscr{A}_X)$ $(X \in L)$. If $X \leqslant Y$ then $\mathscr{A}_X \subseteq \mathscr{A}_Y$. Thus $D^p(\mathscr{A}_X) \supseteq D^p(\mathscr{A}_Y)$ by (2.1). This inclusion map is homogeneous of degree zero. Write

$$D^p(X) = D^p(\mathscr{A}_X). \tag{6.5}$$

The correspondence $X \mapsto D^p(X)$ defines a contravariant functor from $L$ to $\mathscr{C}$ which is also denoted by $D^p$ by abuse of notation.

(6.6) Proposition. *The contravariant functor $D^p$ is local for $p \geqslant 0$.*

*Proof.* Let $\mathfrak{p} \in \operatorname{Spec} S$ and $X \in L$. Write $T = T(\mathfrak{p})$. Localize the inclusion map $D^p(X) \subsetneq D^p(X \wedge T)$ at $\mathfrak{p}$:

$$D^p(X)_{\mathfrak{p}} \to D^p(X \wedge T)_{\mathfrak{p}}.$$

This is injective by the flatness of localization. To show the surjectivity let $\theta \in D^p(X \wedge T) = D^p(\mathscr{A}_{X \wedge T})$. Let $R$ be a defining equation for $\mathscr{A}_X \backslash \mathscr{A}_{X \wedge T}$. Let $\alpha \in V^*$ be a linear factor of $R$. Then $H = \ker(\alpha) \in \mathscr{A}_X \backslash \mathscr{A}_{X \wedge T}$. If $\alpha \in \mathfrak{p}$, then $H \supseteq T$. This shows $H \supseteq X \wedge T$, which is a contradiction. So $\alpha \notin \mathfrak{p}$. Thus $R \notin \mathfrak{p}$. Therefore $R\theta/R \in D^p(X)_{\mathfrak{p}} = D^p(\mathscr{A}_X)_{\mathfrak{p}}$. Its image is $\theta$. This proves the surjectivity. ∎

(6.7) DEFINITION. A contravariant functor $F: L \to \mathscr{C}$ is said to be *cumulative* if for any $X \in L$

$$\sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(F(Y); x)$$

has a pole of order at most $\dim X$ at $x = 1$. In other words, $(x-1)^{\dim X} \sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(F(Y); x)$ is a polynomial for any $X \in L$.

Our aim is to prove the functor $D^p$ is cumulative. This is proved in (6.13).

(6.8) DEFINITION. Let $k$ be a non-negative integer. A contravariant functor $F: L \to \mathscr{C}$ is said to be *$k$-codimensional* if $F(X) = 0$ for any $X \in L$ with $\operatorname{codim} X < k$. We agree that every $F$ is 0-codimensional.

The zero functor is the unique $(l+1)$-codimensional functor. Henceforth we write $\operatorname{Poin}(F(X)) = \operatorname{Poin}(F(X); x)$.

(6.9) PROPOSITION. *Let $F: L \to \mathscr{C}$ be a local and $k$-codimensional contravariant functor. Then, for any $X \in L$,* $\operatorname{Poin}(F(X))$ *has pole at $x = 1$ of order at most $l - k$. In other words, for any $X \in L$, $(x-1)^{l-k}$* $\operatorname{Poin}(F(X))$ *is a polynomial.*

*Proof.* Since the order of the pole of $\operatorname{Poin}(F(X))$ is equal to the dimension $\dim F(X)$ of $F(X)$ as an $S$-module [5, Theorem 19, p. 346], it is sufficient to show that $\dim F(X) \geqslant l - k$. By [4, p. 92, Corollary 3; p. 16, Exercise 2], we have

$$\begin{aligned}\dim F(X) = \text{Krull dim } (S/\operatorname{Ann}(F(X)) &= \max_{\substack{\mathfrak{p} \in \operatorname{Spec} S \\ \mathfrak{p} \supseteq \operatorname{Ann} F(X)}} \text{Krull dim } (S/\mathfrak{p}) \\ &= l - \min_{\substack{\mathfrak{p} \in \operatorname{Spec} S \\ F(X)_{\mathfrak{p}} \neq 0}} \operatorname{ht} \mathfrak{p}.\end{aligned}$$

Thus we only need to prove that $F(X)_{\mathfrak{p}} = 0$ if $\mathfrak{p} \in \operatorname{Spec} S$ and $\operatorname{ht} \mathfrak{p} < k$. Since $\operatorname{codim}(X \wedge T(\mathfrak{p})) \leqslant \operatorname{codim} T(\mathfrak{p}) \leqslant \operatorname{ht} \mathfrak{p} < k$, we have

$$\nu_{\mathfrak{p}}: F(X)_{\mathfrak{p}} \cong F(X \wedge T(\mathfrak{p}))_{\mathfrak{p}} = 0. \quad \blacksquare$$

Let $X \in L$ and $k \geqslant 0$. Let $L_X(k) = \{Z \in X \mid Z \leqslant X \text{ and } \operatorname{codim} X = k\}$.

(6.10) PROPOSITION. *If $F$ is local then $F$ is cumulative.*

*Proof.* Let $F$ be $k$-codimensional $(0 \leqslant k \leqslant l+1)$. We use a descending induction on $k$. When $k = l+1$, $F = 0$. This is trivially cumulative. Let $k \leqslant l$.

We define two contravariant functors $K$ and $C$. For any $X \in L$, define $K(X)$ and $C(X)$ by an exact sequence

$$0 \to K(X) \to F(X) \xrightarrow{\varphi(X)} \bigoplus_{Z \in L_X(k)} F(Z) \to C(X) \to 0. \tag{6.11}$$

The middle map $\varphi(X)$ is the direct product of the natural maps $\nu_{X,Z}$: $F(X) \to F(Z)$ $(Z \in L_X(k))$. Since $\mathscr{C}$ is an abelian category, $K(X)$ and $C(X)$ are both finitely generated graded $S$-modules. Let $X, Y \in L$ with $Y \leqslant X$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K(X) & \longrightarrow & F(X) & \xrightarrow{\varphi(X)} & \bigoplus_{Z \in L_X(k)} F(Z) & \longrightarrow & C(X) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(Y) & \longrightarrow & F(Y) & \xrightarrow{\varphi(Y)} & \bigoplus_{W \in L_Y(k)} F(W) & \longrightarrow & C(Y) \longrightarrow 0.
\end{array} \tag{6.12}$$

The horizontal sequences are the exact sequences (6.11). The left vertical map is the natural map $\nu_{X,Y}: F(X) \to F(Y)$. The right vertical map is obtained in the following way: since $L_X(k) \supseteq L_Y(k)$, $\bigoplus_{W \in L_Y(k)} F(W)$ is a direct summand of $\bigoplus_{Z \in L_X(k)} F(Z)$. The natural projection: $\bigoplus_{Z \in L_X(k)} F(Z) \to \bigoplus_{W \in L_Y(k)} F(W)$ is the right vertical map. Therefore we can define $S$-linear maps $K(X) \to K(Y)$ and $C(X) \to C(Y)$ so that the diagram (6.12) is commutative. They are homogeneous of degree zero. It is easy to show that both $K$ and $C$ satisfy the axioms for a functor. Thus they are contravariant functors from $L$ to $\mathscr{C}$.

*$K$ and $C$ are $(k+1)$-codimensional*: If codim $X < k$, then $F(X) = 0$ and $L_X(k) = \varnothing$. Thus $K(X) = C(X) = 0$. If codim $X = k$, then $L_X(k) = \{X\}$. Thus the map $\varphi(X)$ in (6.11) is an isomorphism. Therefore $K(X) = C(X) = 0$. The functors $K$ and $C$ are thus $(k+1)$-codimensional.

*$K$ and $C$ are local*: Let $\mathfrak{p} \in \operatorname{Spec} S$. Write $T = T(\mathfrak{p})$. Replace $Y$ by $X \wedge T$ in (6.12) and localize at $\mathfrak{p}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & K(X)_{\mathfrak{p}} & \to & F(X)_{\mathfrak{p}} & \to & \bigoplus_{Z \in L_X(k)} F(Z)_{\mathfrak{p}} & \to & C(X)_{\mathfrak{p}} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K(X \wedge T)_{\mathfrak{p}} & \to & F(X \wedge T)_{\mathfrak{p}} & \to & \bigoplus_{W \in L_{X \wedge T}(k)} F(W)_{\mathfrak{p}} & \to & C(X \wedge T)_{\mathfrak{p}} & \to 0.
\end{array}$$

Note that the both rows are exact because they are localizations of exact sequences. The second left vertical map is an isomorphism because $F$ is local. Let $Z \in L_X(k) \backslash L_{X \wedge T}(k)$. Then $\operatorname{codim}(Z \wedge T) < \operatorname{codim} Z = k$. Thus

$F(Z)_{\mathfrak{p}} \cong F(Z \wedge T)_{\mathfrak{p}} = 0$ because $F$ is local and $k$-codimensional. This implies that the second right vertical map is also an isomorphism. Therefore the other two vertical maps are also isomorphisms. In other words the functors $K$ and $C$ are both local.

Thus by the induction assumption both $K$ and $C$ are cumulative. Thanks to the exactness of the second row of (6.12), we obtain

$$\operatorname{Poin}(F(Y)) = \operatorname{Poin}(K(Y)) + \sum_{W \in L_Y(k)} \operatorname{Poin}(F(W)) - \operatorname{Poin}(C(Y)).$$

Fix $X \in L$. Then

$$\begin{aligned}
&(x-1)^{\dim X} \sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(F(Y)) \\
&\quad = (x-1)^{\dim X} \sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(K(Y)) \\
&\qquad + (x-1)^{\dim X} \sum_{Y \in L_X} \mu(Y, X) \sum_{W \in L_Y(k)} \operatorname{Poin}(F(W)) \\
&\qquad - (x-1)^{\dim X} \sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(C(Y)).
\end{aligned}$$

The terms involving $K$ and $C$ are polynomials because $K$ and $C$ are both cumulative. The remaining term is

$$\begin{aligned}
&(x-1)^{\dim X} \sum_{W \in L_X(k)} \operatorname{Poin}(F(W)) \sum_{\substack{Y \in L \\ W \leqslant Y \leqslant X}} \mu(Y, X) \\
&\quad = \begin{cases} (x-1)^{\dim X} \operatorname{Poin}(F(X)) = (x-1)^{l-k} \operatorname{Poin}(F(X)) & \text{if } \operatorname{codim} X = k, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

by (6.1). By (6.9), this term is also a polynomial. This shows that the functor $F$ is cumulative. Thus the induction proceeds. ∎

Recall the definition of $D^p(X)$ in (6.5).

(6.13) Corollary. *The contravariant functor $D^p$ is cumulative for $p \geqslant 0$.*

*Proof.* This follows from (6.6) and (6.10). ∎

Recall the definitions (6.3) and (2.7) of the map $G\colon L \to \mathscr{C}$:

$$\begin{aligned}
G(X) &= (-1)^l \, \Psi(\mathscr{A}_X; 1, t) \\
&= (-1)^l \sum_{p=0}^{l} \operatorname{Poin}(D^p(X); x)(t(x-1)-1)^p \big|_{x=1}.
\end{aligned}$$

In order to complete the proof of the main Theorem (1.4), which asserts $G(X) = \chi(\mathscr{A}_X; t)$, we need only verify the fourth condition of (6.2):

(6.14) PROPOSITION. *The degree in* $t$ *of* $\sum_{Y \in L_X} \mu(Y, X)\, G(Y)$ *does not exceed* $\dim X$ *for all* $X \in L$.

*Proof.* Fix $X \in L$. We compute

$$\begin{aligned}
&\sum_{Y \in L_X} \mu(Y, X)\, G(Y) \\
&\quad = \sum_{Y \in L_X} \mu(Y, X)(-1)^l\, \Psi(\mathscr{A}_Y; 1, t) \\
&\quad = (-1)^l \sum_{p=0}^{l} \sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(D^p(Y); x)(t(x-1)-1)^p|_{x=1} \\
&\quad = (-1)^l \sum_{p=0}^{l} M_p(x)(t(x-1)-1)^p|_{x=1}
\end{aligned}$$

where $M_p(x)$ stands for $\Sigma_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(D^p(Y); x)$. Since the functor $D^p$ is cumulative by (6.13), $(x-1)^{\dim X} M_p(x)$ is a polynomial. Thus the coefficient of $t^n$ in

$$M_p(x)(t(x-1)-1)^p$$

is $(-1)^{p-n}\binom{p}{n} M_p(x)(x-1)^n$, which lies in $(x-1)\, \mathbf{Q}[x]$ if $n > \dim X$. Thus, for each $p$, the degree in $t$ of

$$M_p(x)(t(x-1)-1)^p|_{x=1}$$

does not exceed $\dim X$. Therefore $\sum_{Y \in L_X} \mu(Y, X)\, G(Y)$ has the same property. ∎

## REFERENCES

1. N. BOURBAKI, "Algèbre," Chapitre 3, nouvelle édition, Hermann, Paris, 1971.
2. N. BOURBAKI, "Groupes et algèbres de Lie," Chapitres 4–6, Masson, Paris, 1981.
3. R. HARTSHORNE, "Algebraic Geometry" (R. Hartshorne, Ed.), Graduate Texts in Mathematics Vol. 52, Springer-Verlag, New York/Heidelberg/Berlin, 1977.
4. H. MATSUMURA, "Commutative Algebra," 2nd ed., Benjamin, New York, 1980.
5. G. D. NORTHCOTT, "Lessons on Rings, Modules and Multiplicities," Cambridge Univ. Press, London/New York, 1968.
6. P. ORLIK AND L. SOLOMON, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.
7. P. ORLIK AND L. SOLOMON, Unitary reflection groups and cohomology, *Invent. Math.* **59** (1980), 77–94.

8. K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.
9. R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth & Brooks/Cole, Monterey, CA, 1986.
10. H. Terao, Free arrangements and unitary reflection groups, *Proc. Japan Acad. Ser. A. Math. Sci.* **56**(1980), 389–392.
11. H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard–Todd–Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.
12. H. Terao, Free arrangements of hyperplanes over an arbitrary field, *Proc. Japan Acad. Ser. A Math. Sci.* **59** (1983), 301–303.
13. B. L. van der Waerden, "Modern Algebra," Ungar, New York, 1950.
14. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch. Verw. Gebiete* **2** (1964), 340–368.
15. D. Kelly and G.-C. Rota, Some problems in combinatorial geometry, *in* "A Survey of Combinatorial Theory" (J. N. Srivastava, *et al.*, Eds.), pp. 309–312, North-Holland, Amsterdam, 1973.